M.A./M.Sc. (Mathematics) Semester – III

Paper – XI (Functional Analysis)

Banach Spaces and Examples.

Q. Define a Banach ~~metric~~ space and give its at least two examples.

Ans.- Banach Space :- A nls which is complete w.r.t. the metric generated by its norm, is called a Banach Space.

Examples ① The real linear space R is a Banach Space w.r.t. the norm defined by $||x|| = |x|$ for all $x \in R$.

Verification :- Clearly, $||x|| = |x| \geq 0$ Also, $||x|| = 0$ iff $|x| = 0$ iff $x = 0$, Also, $||\lambda x|| = |\lambda x| = |\lambda| \cdot |x|$ for all $\lambda, x \in R$,

$||x + y|| = |x + y| \leq |x| + |y| = ||x|| + ||y||$ for all $x, y \in R$.

then, $(\mathbb{R}, d)$ is a nls the metric and generated by the norm is given by

$$d(x,y) = \|x-y\| = |x-y| \text{ for all } x, y \in \mathbb{R}.$$

then, $(\mathbb{R}, d)$ is a real Complete metric space, So $\mathbb{R}$ is a real Banach space.

Example②:- The Complex linear space $\mathbb{C}$ of all Complex numbers is a Banach space w.r.t. the norm defined by,

$$\|z\| = |z| \text{ for all } z \in \mathbb{C}.$$

Verification: Clearly, $\|z\| = |z| \geq 0$ $\|z\| = 0$ iff $|z| = 0$ iff $z = 0$.

Also, $\|\alpha z\| = |\alpha \cdot z| = |\alpha| \cdot |z\| = |\alpha| \cdot \|z\|$ for all $\alpha, z \in \mathbb{C}$.

Finally, $\|z+w\| = |z+w| \leq + |w| = \|z\| + \|w\|$ for all $z, w \in \mathbb{C}$.

$\therefore \mathbb{C}$ is nls the metric d generated by the norm is given by $d(z,w) = \|z-w\| = |z-w|$ for all $z, w \in \mathbb{C}$.

We show that, $(\mathbb{C}, d)$ is a Complete metric space.

Let $\{z_n\}$ be any Cauchy sequence in $\mathbb{C}$. then $d(z_m, z_n) = \|z_m - z_n\| = |z_m - z_n| \to 0$ as $m, n \to \infty$.

Now, let $z_n = x_n + i y_n$ for $n = 1, 2, 3 \text{----}$

then $|x_m - x_n| \leq |z_m - z_n| \to 0$ as $m, n \to \infty$.

$$|y_m - y_n| \leq |z_m - z_n| \to 0 \text{ as } m, n \to \infty$$

then, both $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences of real numbers. Since $\mathbb{R}$ is complete, there exists $x, y \in \mathbb{R}$ such that $x_n \to x$ & $y_n \to y$ as $n \to \infty$ i.e. $|x_n - x| \to 0$ as $n \to \infty$.

Let $z = x + iy$, then $z \in \mathbb{C}$.

Also, $(z_n, z) = \|z_n - z\| = |z_n - z|$

$= |x_n + iy_n - (x + iy)| = |(x_n - x) + i(y_n - y)|$

$\leq |x_n - x| + |i(y_n - y)|$

$\therefore d(z_n, z) = |x_n - x| + |y_n - y| \to 0$ as $n \to \infty$.

$\therefore \{z_n\}$ converges to $z \in \mathbb{C}$. Hence $(\mathbb{C}, d)$ is a complete metric space w for all $z, w \in \mathbb{C}$.

$\therefore \mathbb{C}$ is a Banach space.

Example:- The set (space) $\mathbb{R}^n$ of all n-tuples $(x_1, x_2, ---- x_n)$ of real numbers is a Banach space w.r.t. addition, scalar multiplication are norm defind as follows:

For every $x = (x_1, ----, x_n)$, $y = (y_1, ---- y_n)$ in $\mathbb{R}^n$, we define $x + y = (x_1 + y_1, ----, x_n + y_n)$ and for every $\lambda \in K$, we define $\lambda x = (\lambda x_1, ---- \lambda x_n)$. $\|x\| = \sqrt{\sum_{i=1}^{n} x_i^2}$.

Verification:- $\mathbb{R}^n$ is clearly a linear space over the field K of scalar w.r.t. the above definition of addition and scalar multiplication,

Now, $\|x\| = \sqrt{\sum_{i=1}^{n} x_i^2} \geq 0$, Also $\|x\| = 0$

iff, $\sqrt{\sum_{i=1}^{n} x_i^2} = 0$ iff $\sum_{i=1}^{n} x_i^2 = 0$ iff $x_i^2 = 0$

for $i = 1, ---- n$ iff $x_i = 0$ for $i = 1, ---- n$, iff $x = 0$, Also, for any $x \in \mathbb{R}^n$ and any $\lambda \in K$

$$\|\lambda x\| = \sqrt{\sum_{i=1}^{n} (\lambda x_i)^2} = |\lambda| \sqrt{\sum_{i=1}^{n} x_i^2} = |\lambda| \cdot \|x\|.$$

Again, in order to verify the fourth condition, we need Cauchy Schwarz inequality For, $(x_1, x_2 ---- x_n)$ & $(y_1, y_2, ---- y_n)$ in $\mathbb{R}^n$,

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right)$$

i.e. $\sum_{i=1}^{n} x_i y_i \leq \|x\| \cdot \|y\|$

Now, $\|x+y\|^2 = \sum_{i=1}^{n} (x_i + y_i)^2$

$$= \sum_{i=1}^{n} (x_i^2 + 2x_i y_i + y_i^2)$$

$$= \sum_{i=1}^{n} x_i^2 + 2\sum_{i=1}^{n} x_i y_i + \sum_{i=1}^{n} y_i^2 \leq \|x\|^2$$

$$+ 2\|x\|\cdot\|y\| + \|y\|^2$$

i.e. $\|x+y\|^2 \leq (\|x\| + \|y\|)^2$

$\therefore \|x+y\| \leq \|x\| + \|y\|$

$\therefore \mathbb{R}^n$ is a normed linear space the metric d generated by the norm is given by,

$$d(x,y) = \|x-y\| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

It remains to show that $(\mathbb{R}^n, d)$ is a complete metric space.

Let $\{x^{(m)}\}$ be any Cauchy sequence in $\mathbb{R}^n$. Where $x^{(m)} = (x_1^{(m)}, \dots, x_n^{(m)})$ then $d(x^{(m)}, x^{(p)}) \to 0$

$d(x^{(m)}, x^{(p)}) \to 0$ as $m, p \to \infty$ Now, for each fixed i

i.e. $|x_i(m) - x_i(p)| \leq \sqrt{\sum_{i=1}^{n} (x_i(m) - x_i(p))^2}$

$$= d(x^{(m)}, x^{(p)}) \to 0 \text{ as, } m, p \to \infty.$$

$\therefore$ For each fixed i, $\{x_i(m)\}$ is a Cauchy sequence of real numbers.

So, there exist, $x_i \in R$ such that $x_i^{(m)} \to x_i$ as $m \to \infty$

$\therefore$ given $\epsilon > 0$, there exists a +ve integer number such that

$$|x_i^{(m)} - x_i| < \frac{\epsilon}{\sqrt{n}} \text{ for all } m \geq n_0$$

Let $x = (x_1, \ldots x_n)$, then $x \in R^n$

Now, $d(x^{(m)}, x) = \sqrt{\sum_{i=1}^{n} (x_i^{(m)} - x_i)^2} < \epsilon$ for all $m \geq n_0$

Hence, $x^{(m)} \to x$ as $m \to \alpha$ i.e. (R^n, d) is complete.

$\therefore R^n$ is a Banach space.